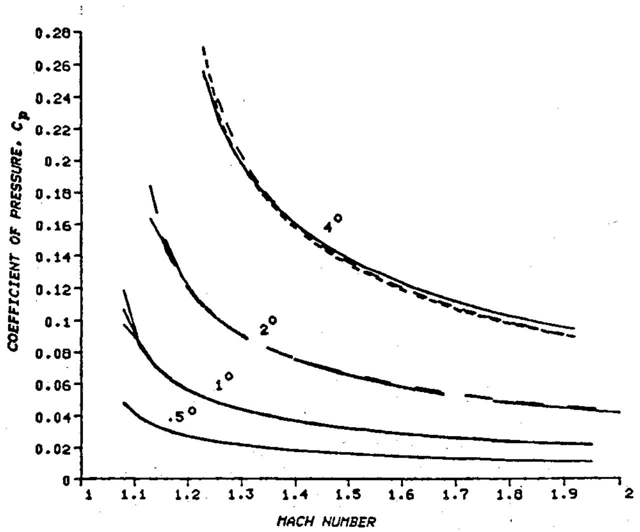

Fig. 2 Solid curve gives pressure coefficient as compiled from oblique shock theory. Short dashed curve as given by Eq. (17); long dashed curve as given by Eq. (18).

Eqs. (13) and (15) is significantly better. In the latter case the departure for $0.8 \leq M \leq 1.1$ is only a fraction of 1%. In view of the excellent agreement with transonic data found with Eq. (15) (Murman[6]), similar agreement should also be found using Eq. (13).

Returning to Eq. (11) and neglecting the higher order terms contained in the curly brackets, we may integrate to obtain

$$v = \frac{2}{3\alpha}\left[\beta^3 - (\beta^2 + \alpha u)^{3/2}\right] \tag{16}$$

The negative branch of the square root was taken, corresponding to upper half plane flow.

An immediate application of Eq. (16) is the computation of the pressure coefficient at a body. The pressure coefficient has the form $C_p = -2u$ in the present approximation; thus solving Eq. (16) and substituting we obtain

$$C_p = \frac{2}{\alpha}\left[\beta^2 - (\beta^3 - \frac{3}{2}\alpha\theta)^{2/3}\right] \tag{17}$$

where we have introduced the linearized boundary condition $v = \theta\,[\theta = \tan^{-1}(f'(x))$; $f(x)$ the body shape).

Spreiter and Alksne[8] suggest that the pressure coefficient for transonic flow in the low supersonic limit is

$$C_{SA} = \frac{2}{(\gamma+1)M^2}\left\{\beta^2 - [\beta^3 - \frac{3}{2}(\gamma+1)M^2\theta]^{2/3}\right\} \tag{18}$$

It is convenient to discuss these results in terms of flow past a wedge, in which case an exact result for the pressure coefficient is known from oblique shock theory. Figure 2 contains plots of the pressure coefficient at the four small angles of 0.5°, 1°, 2°, and 4°, and in each case exact oblique shock theory as well as Eqs. (17) and (18) are shown. In each case Eq. (17) lies closer than Eq. (18) to oblique shock theory, however the numerical differences are quite small.

One further point to note in Fig. 2 is that in the case of a 4° wedge both Eqs. (17) and (18) begin to depart seriously from the exact theory at higher Mach numbers. This might have been anticipated since neither Eq. (12) nor Eq. (1) with $\Gamma = (\gamma+1)M^2$ are appropriate at supersonic speeds. Under this limit Hayes' equation, i.e., Eq. (1) with $\Gamma = (\gamma+1)M^4$ is more appropriate. In fact performing the same analysis on the Hayes equation we obtain

$$C_p = \frac{2}{(\gamma+1)M^4}\left\{\beta^2 - [\beta^3 - \frac{3}{2}(\gamma+1)M^4\theta]^{2/3}\right\}$$

which under the supersonic limit $K \to \infty$, or equivalently θ small, becomes

$$C_p \sim \frac{2}{\beta}\theta + \frac{\gamma M^4 + (M^2-2)^2}{2\beta^4}\theta^2$$

This will be recognized as the Busemann second-order expansion (Busemann[9], Lighthill[10]). Therefore, said in other terms the discrepancy of Eqs. (17) and (18) with exact theory at higher Mach numbers is due to the fact that both of these are only valid to first order in that limit.

References

[1] von Karman, R., "The Similarity Law of Transonic Flow," *Journal of Mathematical Physics*, Vol. 26, Oct. 1947, pp. 182-190.

[2] Spreiter, J. R., "On the Application of Transonic Similarity Rules to Wings of Finite Span," NACA Tech. Rpt. 1153, 1953.

[3] Hayes, W. D., "La seconde approximation pour les ecoulements transsoniques non visqueux," *Journal de Mechanique*, Vol. 5, June 1966, pp. 163-206, (Errata pp. 397-398).

[4] Hayes, W. D., "Pseudotransonic Similitude and First-Order Wave Structure," *Journal of Aerospace Science*, Vol. 11, Nov., 1954, pp. 721-730.

[5] Murman, E. M. and Krupp, J. A., "Solution of the Transonic Potential Equation Using a Mixed Finite Difference System," *Lecture Notes in Physics*, Vol. 8, Springer-Verlag, Berlin, 1971, pp. 199-206.

[6] Murman, E. M., "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," *Proceedings of the AIAA Computational Fluid Dynamics Conference*, Palm Springs, Calif., July 1973, pp. 27-40.

[7] Cole, J. D., "Twenty Years of Transonic Flow," Document D1-82-0878, Boeing Scientific Research Laboratories, Seattle, Washington, 1969, Invited Lecture presented at AIAA Meeting in San Francisco, June 1969.

[8] Spreiter, J. R. and Alksne, A. Y., "Thin Airfoil Theory Based on Approximate Solution of the Transonic Flow Equations," NACA Rept. 1359, 1958.

[9] Busemann, A., "Aerodynamischer Auftrieb bei Uberschallgeschwindigkeit," *Luftfahrtforschung*, Vol. 12, 1935, p. 210.

[10] Lighthill, M. J., *Higher Approximations in Aerodynamic Theory*, Princeton University Press, 1954, p. 44.

Laminar Compressible Boundary Layers with Vectored Mass Transfer

G. Nath* and Margaret Muthanna†
Indian Institute of Science, Bangalore, India

Introduction

THE study of the effects of surface mass transfer involving both normal and tangential components (vectored suction or injection) on bodies is useful in understanding several important boundary-layer problems such as transpiration cooling of rocket engines and turbine blades, mixing processes in pollution problems, flow on reentry bodies, and many other similar problems.[1] Inger and Swean,[1] and Scala and Sutton[2] have studied the effects of vectored suction or injection on laminar compressible boundary-layer flow of a gas with constant properties (i.e., $\rho\mu$ = constant, $Pr = 1$ and 0.7, where ρ, μ, and Pr are the density, viscosity, and Prandtl number, respectively) at constant pressure, and at an axisymmetric stagnation point, respectively.

In the present study, we have extended the previous analysis of Ref. 2 to include the effects of variable gas properties. In particular, we have studied the effects of vectored suction or

Received Feb. 25, 1976; revision received April 5, 1976.
Index category: Boundary Layers and Convective Heat Transfer-Laminar.

*Associate Professor, Department of Applied Mathematics.
†Research Student, Department of Applied Mathematics.

injection on the flow and heat transfer at the stagnation point of a two-dimensional body (a cylinder) and an axisymmetric body (a sphere).

Basic Equations

The boundary-layer equations in dimensionless form for the steady compressible fluid with variable properties ($\rho \alpha T^{-1}$, $\mu \alpha T^{\omega}$, $Pr=0.72$, where T and ω are temperature and exponent in the power-law variation of viscosity, respectively) in the stagnation region of a two-dimensional or an axisymmetric body with tangential and normal surface mass transfer under similarity requirements can be expressed as[3,4]

$$(cf'')' + ff'' + \beta(g - f'^2) = 0 \quad (1)$$

$$(cg')' + Prfg' = 0 \quad (2)$$

The boundary conditions are

$$f(0) = f_w, f'(0) = f'_w, f'(\infty) = 1; g(0) = g_w, g(\infty) = 1 \quad (3)$$

where f' and g are the dimensionless velocity and total enthalpy, respectively; $c = \rho\mu/\rho_e\mu_e = g^{\omega-1}$ is the ratio of the product of viscosity and density; β is the pressure-gradient parameter ($\beta=1$ for the cylinder and $\beta=0.5$ for the sphere);

$$f_w = -(\rho v)_w (2\xi)^{1/2} / \rho_e \mu_e u_e r_0^j$$

and

$$f'_w = u_w / u_e$$

are the normal and tangential surface mass transfer parameters, respectively (for downstream vectored injection: $f'_w > 0$, $f_w < 0$; for downstream vectored suction: $f'_w > 0$, $f_w > 0$; for upstream vectored injection: $f'_w < 0$, $f_w < 0$; for upstream vectored suction: $f'_w < 0$, $f_w > 0$) the index $j=0$ for the cylinder and $J=1$ for the sphere; g_w is the dimensionless wall temperature; u and v are the velocity components along and perpendicular to the surface of the body;

$$\xi = \int_0^x \rho_e \mu_e u_e r_0^{2j} dx$$

is the transformed coordinate; prime denotes differentiation with respect to the similarity variable

$$\eta(\eta = u_e (2\xi)^{-1/2} \int_0^y r_0^j \rho dy$$

and subscripts w and e denote conditions at the wall and at the edge of the boundary layer, respectively.

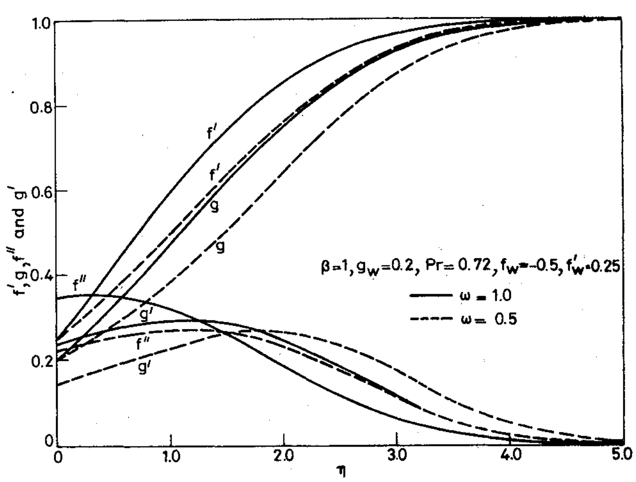

Fig. 1 Velocity, total enthalpy, skin-friction, and heat-transfer profiles.

It may be noted that the solution of Eqs. (1) and (2) in the absence of vectored suction or injection (i.e., for $f_w = f'_w = 0$) has been obtained by Gross and Dewey.[3] We also note that $\omega = 0.5$ represents conditions encountered in hypersonic flows, $\omega = 0.7$ corresponds closely to low-temperature flows, whereas $\omega = 1$ corresponds to the familiar assumption of constant density-viscosity product.

The skin-friction coefficient C_f and the heat-transfer coefficient in terms of the Stanton number St can be expressed in the form

$$C_f = 2\tau_w / \rho_e u_e^2 = 2^{3/2} (Re_x)^{-1/2} F''_w \quad (4)$$

$$St = q_w / [(I_e - I_w)\rho_e u_e] = 2^{1/2} (Re_x)^{-1/2} G'_w \quad (5)$$

where $F''_w = g_w^{\omega-1} f''_w$ and $G'_w = g_w^{\omega-1} g'_w / [Pr(1-g_w)]$ are skin-friction and heat-trnasfer parameters respectively, $Re_x = (x^2/v_e)(du_e/dx)$ is the local Reynolds number, and τ_w and q_w are the surface shear stress and surface heat-transfer rate, respectively.

Results and Discussion

Equations (1) and (2) under conditions (3) have been solved numerically using quasilinearization technique[4] on IBM 360/44 computer for both cylinder ($\beta = 1$) and sphere ($\beta = 0.5$) for various values of ω, f_w, f'_w, and $g_w (0 < g_w < 1)$. Although, computations were carried out for 294 conditions involving several parameters, for the sake of brevity, only some representative velocity, total enthalpy, skin friction and heat-transfer profiles for $g_w = 0.2$, $\beta = 1$, $f_w = 0.5$ and, $f'_w = 0.25$ are shown in Fig. 1. It is observed from Fig. 1 that for the downstream vectored injection ($f_w = -0.5$, $f'_w = 0.25$) there is a point of inflexion in the velocity and total enthalpy profiles for all values of ω when $g_w = 0.2$ as is evident from the maximum in f'' and g' for certain value of η. On the other hand, when $f_w = 0$, the velocity and total enthalpy profiles do not have a point of inflexion for $\omega = 1$ whatever may be the value of f'_w ($f'_w \geq 0$). Furthermore, when $g_w = 0.6$, $f_w = 0.5$, and $f'_w = 0$, the velocity and total enthalpy profiles do not have a point of inflexion for any value of ω. It may be remarked that the above results also hold good for a sphere ($\beta = 0.5$). It may be noted that the existence of a point of inflexion implies that the flow is prone to instability but suction ($f_w > 0$) and large values of wall temperature (i.e., $g_w \geq 0.6$) tend to stabilize the flow.

The skin-friction parameter F''_w and the heat-transfer parameter G'_w for downstream injection and suction ($f_w \lesseqgtr 0$, $f'_w > 0$) and also for zero mass transfer ($f_w = f'_w = 0$) for various values of ω, β and g_w are given in Tables 1 and 2. The upstream injection results have also been obtained, but they are not given here for lack of space.‡ It may be remarked that upstream injection can occur somewhat downstream from the stagnation point, but it cannot occur exactly at the stagnation point. It is observed that for all values of ω, g_w and, f_w, F''_w decreases but G'_w increases as f'_w increases. It is also seen that F''_w and G'_w increase as ω decreases or β, or g_w increases whatever may be the values of f_w and f'_w. However, for $f'_w = 0.5$, F''_w and G'_w change little as β increases from 0.5 to 1. It may be noted that for a given $g_w (0 < g_w < 1)$ the parameters f''_w and g'_w which occur in skin-friction and heat-transfer coefficients [see Eqs. (4) and (5)] decrease as ω decreases, but $g_w^{\omega-1}$ increases as ω decreases. Consequently, as mentioned earlier, F''_w and G'_w increase as ω decreases. We have compared our results for $\beta = 0$, $\omega = Pr = 1$ (results are not tabulated here) with those tabulated by Inger and Swean and they are found to be in excellent agreement. However, the results for $\beta = 0.5$ and $\omega = 1$ could not be compared with those of Scala and Sutton as they were not available in tabular form.

‡The results can be obtained from the authors.

Table 1 Skin-friction and heat-transfer parameters for $Pr = 0.72$ and $\beta = 1$ (cylinder)

g_w	f_w	f'_w	$\omega = 1.0$		$\omega = 0.7$		$\omega = 0.5$	
			F''_w	G'_w	F''_w		F''_w	G'_w
0.2	0	0	0.7464	0.6335	0.8478	0.6947	0.9266	0.7411
0.2	0	0.25	0.5605	0.7129	0.6381	0.7927	0.6987	0.8539
0.2	0	0.50	0.3084	0.7827	0.3458	0.8781	0.3745	0.9507
0.2	0.25	0	0.9157	0.8123	1.0115	0.8699	1.0865	0.9138
0.2	0.25	0.25	0.6842	0.8877	0.7579	0.9633	0.8159	1.0217
0.2	0.25	0.50	0.3908	0.9557	0.4271	1.0467	0.4553	1.1173
0.2	0.50	0	1.0969	1.0029	1.1865	1.0564	1.2372	1.0974
0.2	0.50	0.25	0.8172	1.0739	0.8869	1.1451	0.9419	1.2004
0.2	0.50	0.50	0.4798	1.1388	0.5144	1.2252	0.5414	1.2928
0.2	−0.25	0	0.5915	0.4695	0.6989	0.5349	0.7823	0.5843
0.2	−0.25	0.25	0.4478	0.5521	0.5285	0.6353	0.5910	0.6986
0.2	−0.25	0.50	0.2338	0.6247	0.2721	0.7228	0.3016	0.7983
0.2	−0.50	0	0.4538	0.3243	0.5654	0.3920	0.6521	0.4430
0.2	−0.50	0.25	0.3479	0.4080	0.4304	0.4930	0.4943	0.5577
0.2	−0.50	0.50	0.1680	0.4808	0.2060	0.5807	0.2354	0.6572
0.6	0	0	0.9987	0.6677	1.0545	0.6939	1.0935	0.7118
0.6	0	0.25	0.7865	0.7404	0.8307	0.7714	0.8619	0.7931
0.6	0	0.50	0.5154	0.8066	0.5441	0.8418	0.5643	0.8669
0.6	0.25	0	1.1559	0.8405	1.2086	0.8648	1.2459	0.8819
0.6	0.25	0.25	0.9020	0.9108	0.9443	0.9403	0.9743	0.9610
0.6	0.25	0.50	0.5918	0.9753	0.6195	1.0094	0.6390	1.0333
0.6	0.50	0	1.3247	1.0260	1.3745	1.0486	1.4098	1.0645
0.6	0.50	0.25	1.0262	1.0933	1.0665	1.1211	1.0950	1.1406
0.6	0.50	0.50	0.6741	1.1554	0.7006	1.1878	0.7193	1.2107
0.6	−0.25	0	0.8550	0.5105	0.9129	0.5378	0.9537	0.5568
0.6	−0.25	0.25	0.6807	0.5840	0.7265	0.6164	0.7587	0.6388
0.6	−0.25	0.50	0.4454	0.6509	0.4750	0.6874	0.4958	0.7129
0.6	−0.50	0	0.7262	0.3720	0.7859	0.3998	0.8278	0.4192
0.6	−0.50	0.25	0.5856	0.4445	0.6325	0.4772	0.6654	0.4998
0.6	−0.50	0.50	0.3825	0.5103	0.4126	0.5472	0.4336	0.5729

Table 2 Skin-friction and heat-transfer parameters for $Pr = 0.72$ and $\beta = 0.5$ (sphere)

g_w	f_w	f'_w	$\omega = 1.0$		$\omega = 0.7$		$\omega = 0.5$	
			F''_w	G'_w	F''_w	G'_w	F''_w	G'_w
0.2	0	0	0.6350	0.6154	0.7131	0.6736	0.7732	0.7178
0.2	0	0.25	0.5056	0.7059	0.5718	0.7844	0.6232	0.8447
0.2	0	0.50	0.3160	0.7847	0.3559	0.8793	0.3868	0.9523
0.2	0.25	0	0.8087	0.7979	0.8821	0.8525	0.9391	0.8940
0.2	0.25	0.25	0.6319	0.8819	0.6949	0.9566	0.7439	1.0140
0.2	0.25	0.50	0.3997	0.9568	0.4384	1.0478	0.4684	1.1184
0.2	0.50	0	0.9951	0.9915	1.0635	1.0424	1.1169	1.0811
0.2	0.50	0.25	0.7681	1.0694	0.8275	1.1398	0.8741	1.1941
0.2	0.50	0.50	0.4901	1.1399	0.5269	1.2265	0.5559	1.2943
0.2	−0.25	0	0.4774	0.4474	0.5601	0.5093	0.6238	0.5559
0.2	−0.25	0.25	0.3911	0.5434	0.4598	0.6255	0.5129	0.6879
0.2	−0.25	0.50	0.2403	0.6253	0.2812	0.7237	0.3126	0.7993
0.2	−0.50	0	0.3395	0.2979	0.4256	0.3635	0.4921	0.4134
0.2	−0.50	0.25	0.2907	0.3977	0.3604	0.4817	0.4143	0.5454
0.2	−0.50	0.50	0.1737	0.4812	0.2142	0.5813	0.2457	0.6580
0.6	0	0	0.7858	0.6396	0.8258	0.6637	0.8540	0.6804
0.6	0	0.25	0.6380	0.7239	0.6713	0.7540	0.6948	0.7750
0.6	0	0.50	0.4358	0.7993	0.4587	0.8341	0.4747	0.8588
0.6	0.25	0	0.9498	0.8170	0.9875	0.8394	1.0141	0.8553
0.6	0.25	0.25	0.7580	0.8969	0.7898	0.9252	0.8123	0.9454
0.6	0.25	0.50	0.5150	0.9687	0.5370	1.0025	0.5525	1.0265
0.6	0.50	0	1.1268	1.0066	1.1621	1.0276	1.1870	1.0422
0.6	0.50	0.25	0.8877	1.0816	0.9178	1.1085	0.9390	1.1274
0.6	0.50	0.50	0.6006	1.1500	0.6215	1.1822	0.6363	1.2049
0.6	−0.25	0	0.6376	0.4778	0.6795	0.5031	0.7090	0.5209
0.6	−0.25	0.25	0.5289	0.5653	0.5635	0.5966	0.5878	0.6181
0.6	−0.25	0.50	0.3637	0.6424	0.3872	0.6783	0.4037	0.7038
0.6	−0.50	0	0.5078	0.3351	0.5509	0.3614	0.5812	0.3797
0.6	−0.50	0.25	0.4325	0.4233	0.4678	0.4549	0.4924	0.4769
0.6	−0.50	0.50	0.2995	0.5007	0.3232	0.5371	0.3400	0.5626

Conclusions

The variation of the density-viscosity product across the boundary layer has a significant effect on the skin friction and heat transfer. Also, it gives rise to a point of inflexion which can be removed by suction and by increasing the wall temperature. The skin friction and heat transfer are significantly affected by the pressure gradient parameter.

References

[1]Inger, G.R. and Swean, T.F., "Vectored Injection into Laminar Boundary Layers with Heat Transfer," *AIAA Journal*, Vol. 13, May 1975, pp. 616-622.

[2]Scala, S. and Sutton, G., "Vectored Injection into a Hypersonic Laminar Boundary Layer," *Jet Propulsion*, Vol. 27, Aug. 1957, pp. 895-896.

[3]Gross, J.F. and Dewey, C.F. Jr., "Similar Solutions of the Laminar Boundary Layer Equations with Variable Fluid Properties," *Fluid Dynamics Transactions*, ed, W.Friszdon, Vol. 2, Pergamon Press, New York, 1965, pp. 529-548.

[4]Libby, P.A. and Liu, T.M., "Some Similar Laminar Flows Obtained by Quasilinearization," *AIAA Journal*, Vol. 6, Aug. 1968, pp. 1541-1548.

On Twisting of Orthotropic Plates in a Large Deflection Regime

Ramesh Chandra*

National Aeronautical Laboratory, Bangalore, India

Nomenclature

- a = plate dimension
- h = plate thickness
- w = deflection of a point on median surface of plate in direction normal to the undeformed plate
- E_x, E_y = Young's moduli of plate in x,y directions
- x,y = coordinates in fiber and perpendicular to fiber directions
- G_{xy} = shear modulus of plate in x,y plane
- μ_{xy}, μ_{yx} = Poisson's ratios, first subscript denotes load direction, second denotes lateral direction
- P = applied load
- E = E_x
- F = E_x/E_y
- μ = μ_{xy}
- η = E_x/G_{xy}
- δ = $(F-\mu^2/\eta F$
- ϕ = airy stress function
- w_0 = deflection under the load
- $\bar{w}_s$ = nondimensional deflection at load in small deflection range
- Q_z = transverse shear

Introduction

A PLATE twist test[1] normally is used to determine the shear modulus of an orthotropic plate. One of the requirements for this test to yield reliable results is that the deflection measured should be within a small deflection range. Thick plate specimens are required to insure deflections in the small deflection domain, in view of the other restrictions on the plate dimensions, load applied, and the deflection measured. Thin plate specimens could be used for this test, and, thereby, much material could be saved, provided that the theoretical development existed which included large deflection theory of plates and self weight. To the author's best knowledge, such development does not exist.

Received Nov. 13, 1975; revision received Jan. 22, 1976.

Index category: Structural Composite Materials (including Coatings).

*Scientist, Structural Sciences Division.

Hence, an attempt is made in this paper to understand the large deflection behavior of orthotropic plates subjected to pure twisting moment.

Analysis

The governing equation in stress function, which is derived by combining the compatibility equation and stress-strain relations for orthotropic plate, is reproduced here from Ref. 2 as follows:

$$F\frac{\partial^4\phi}{\partial x4} + (\eta-2\mu)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = E(w,^2_{xy} - w,_{xx}w,_{yy}) \tag{1}$$

Using strain-stress resultant relations and bending moment-curvature relations in the strain-energy expression for orthotropic plates,[3] the following is obtained:

$$U_{ST} = \tfrac{1}{2}\int_0^a\int_0^a\left\{\frac{1}{Eh}(N_x^2 - 2\mu N_xN_y + FN_y^2 + \eta N_{xy}^2) + \frac{EFh^3}{12(F-\mu^2)}\left(w,^2_{xx} + 2\frac{\mu}{F}w,_{xx}w,_{yy} + \frac{1}{F}w,^2_{yy} + 4\delta w,^2_{xy}\right)\right\}dxdy \tag{2}$$

Knowing the deflection w and induced inplane forces, the strain energy in the plate can be computed using this expression.

The square plate is supported at two opposite corners from below and at a third corner from above. The load is applied at the fourth corner. Thus, the plate is subjected to pure twisting moments distributed at all four edges.

The potential V because of applied load and self weight is

$$V = P(w)_{x=0,y=0} + \int_0^a\int_0^a \rho w dxdy \tag{3}$$

Thus, the total potential energy of the plate is as follows:

$$U = U_{ST} - V \tag{4}$$

The boundary conditions for this plate are

at $x=0$ edge,

$$Q_z(0,y)=0, \quad 0<y<a \tag{5a}$$

$$M_x(0,y)=0, \quad 0\leq y\leq a \tag{5b}$$

$$w(0,a)=0 \tag{5c}$$

at $x=a$ edge,

$$Q_z(a,y)=0, \quad 0<y<a \tag{6a}$$

$$M_x(a,y)=0, \quad 0\leq y\leq a \tag{6b}$$

$$w(a,0)=w(a,a)=0 \tag{6c}$$

at $y=0$ edge,

$$Q_z(x,0)=0, \quad 0<x<a \tag{7a}$$

$$M_y(x,0)=0, \quad 0\leq x\leq a \tag{7b}$$

at $y=a$ edge,

$$Q_z(x,a)=0, \quad 0<x<a \tag{8a}$$

$$M_y(x,a)=0, \quad 0\leq x\leq a \tag{8b}$$

The deflection function w satisfying the geometric boundary conditions is assumed as

$$w=A(x-a)(y-a) \tag{9}$$